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Dynamic Condensation Approach for Nonclassically Damped Structures

Mario A. Rivera*
University of Puerto Rico, Mayaguez, Puerto Rico 00681
Mahendra P. Singh†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24041
and
Luis E. Suarez‡
University of Puerto Rico, Mayaguez, Puerto Rico 00681

An iterative dynamic condensation procedure for calculating the eigenproperties of damped dynamic systems is presented. The kept and reduced degrees of freedom of the systems are related through dynamic condensation matrices. The formulation to define the condensation matrices is developed. The procedure starts with a classical static condensation scheme but iteratively improves the results. The complex eigenproperties calculated in an iteration step are used to construct an updated condensation matrix for the following iteration step, which in turn is utilized to form a revised condensed system. A few iterations can provide accurate values of the eigenproperties. The iterative process can be expedited initially by a proper selection of the kept and reduced coordinates. Numerical results are presented to demonstrate the accuracy and convergence characteristics of the proposed condensation scheme.

Nomenclature

$[A]$ = system matrix in state space
$[B]$ = system matrix in state space
$[C]$ = nonclassical damping matrix
$\boldsymbol{F}(t)$ = force vector
$[G]$ = Guyan condensation matrix
$[I]$ = unity matrix
$[K]$ = stiffness matrix
$[M]$ = mass matrix
$\boldsymbol{Q}(t)$ = force vector in state space
$[R]$ = condensation matrix
$[S]$ = transformation matrix
$\boldsymbol{x}$ = displacement vector
$\boldsymbol{y}$ = state-space vector
β = equivalent damping ratio
ϵ = relative error, %
θ = imaginary part of the eigenvalue
λ = complex eigenvalue
σ = real part of the eigenvalue
$[\Psi]$ = complex eigenvector matrix
$[\Omega]$ = complex eigenvalue matrix
ω = equivalent modal frequency
$[\,]^T$ = transpose of a matrix
$[\,]^{-1}$ = inverse of a matrix

Subscripts

h = matrix associated with the condensation process with higher modes
k = kept coordinates when iterating with lower modes
l = matrix associated with the condensation process with lower modes
r = kept coordinates when iterating with higher modes
α = substructure α
β = substructure β

Superscripts

(i) = ith iteration step
(0) = initial iteration step

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*Assistant Professor, Department of General Engineering.
†Professor, Department of Engineering Science and Mechanics.
‡Professor, Department of Civil Engineering. Member AIAA.

Introduction

SINCE the first proposal for condensation of structural systems by Guyan[1] and Irons,[2] there have been many investigations to improve the accuracy of the condensation procedures for dynamic analysis of structures. The literature is replete with papers on this subject. A comprehensive review of this topic is beyond the scope of this paper. An interested reader can refer to the review paper by Paz[3] for earlier contributions in this area.

The majority of dynamic condensation methods proposed in the past have been restricted to the determination of the undamped vibration modes, or the modes that can be used for the analysis of classically damped structures. However, there may be situations in which the classical damping assumptions are invalid. Examples of such cases are the structures made up of materials with different damping characteristics in different parts, structures equipped with passive control systems or with concentrated dampers, structures with layers of damping materials, and structural systems with rotating parts. For a linear dynamic analysis of such structural systems, complex eigenproperties are required to incorporate properly the nonclassical damping effect.

When complex structural systems are discretized with finite elements, the resulting mathematical model may have a very large number of degrees of freedom. In addition, if the energy dissipation of the structural system is modeled using the viscous nonclassically damped formulation and obtaining the linear response of the system using modal analysis is desired, the equations of motion first have to be written in state-space form. Thus, the number of degrees of freedom of the new equations of motion and the size of the associated eigenvalue problem required to obtain the complex eigenvalues and eigenvectors are doubled automatically. To treat these cases, one needs some techniques to reduce the full eigenproblem to a much smaller one and thus lessen the computational effort needed to extract the required eigensolution. The dynamic condensation or dynamic reduction method is one of the techniques that can be applied to extract the lower eigenpairs of large damped systems by solving a small eigenproblem. A few publications dealing with dynamic reduction techniques of damped systems are Refs. 4–7.

In this paper a dynamic condensation approach applicable to non-classically damped structures is developed. This approach is an important generalization and extension of the condensation approach proposed by the last two authors of the present paper, Suarez and Singh, for undamped structures. Like the method by Suarez and Singh[8] for undamped systems, the method proposed in the current paper for damped systems is also iterative. The eigenproperties obtained in an iteration step are used to improve the condensation matrix in the following iterative step. Accurate values can be obtained with a few iteration steps. The formulation also is developed to calculate the remaining eigenproperties from the solution of a complementary eigenvalue problem. Examples are presented to illustrate the effectiveness and convergence properties of the proposed approach. The effect of the selection of coordinates in the condensation process on the accuracy of the eigenproperties also is examined critically.

Formulation of the Method

Eigenvalue Analysis

We consider the case of a viscously damped structural system discretized with N degrees of freedom with equations of motion:

$$[M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{F}(t) \tag{1}$$

We consider a general case in which the damping matrix $[C]$ is not classical, i.e., the normal modes of the undamped structural system cannot be used to uncouple Eq. (1). For the modal analysis of such systems the state vector approach is commonly used. In this approach, using an auxiliary equation, Eq. (1) is transformed into a set of first-order equations of the following form:

$$[A]\dot{\boldsymbol{y}} - [B]\boldsymbol{y} = \boldsymbol{Q}(t) \tag{2}$$

in which $[A]$ and $[B]$ are $(2N \times 2N)$ real symmetric matrices and $\boldsymbol{Q}(t)$ is a $(2N \times 1)$ vector, defined as

$$[A] = \begin{matrix} [0] & [M] \\ [M] & [C] \end{matrix}, \qquad [B] = \begin{matrix} [M] & [0] \\ [0] & -[K] \end{matrix}$$

$$\boldsymbol{Q}(t) = \begin{matrix} \boldsymbol{0} \\ \boldsymbol{F}(t) \end{matrix} \tag{3}$$

Vector $\boldsymbol{y}$ is defined as

$$\boldsymbol{y} = \begin{matrix} \dot{\boldsymbol{x}} \\ \boldsymbol{x} \end{matrix} \tag{4}$$

The eigenproperties of the structural system that uncouple Eq. (2) can be obtained from the solution of its associated eigenvalue problem

$$[A][\Psi][\Omega] = [B][\Psi] \tag{5}$$

in which $[\Omega]$ is a diagonal matrix containing the $2N$ eigenvalues and $[\Psi]$ is the $(2N \times 2N)$ eigenvector matrix. Eigenvalues and corresponding eigenvectors occur in complex conjugate pairs. Here the eigenvectors are assumed to have been normalized such that

$$[\Psi]^T[A][\Psi] = [I], \qquad [\Psi]^T[B][\Psi] = [\Omega] \tag{6}$$

where $[I]$ is a $(2N \times 2N)$ unity matrix.

In general, the dimension of this eigenvalue problem can be large. Instead of solving it directly, we propose to solve two reduced-size eigenvalue problems to obtain the complete set of eigenproperties. To construct the reduced eigenvalue problem, the displacement vector $\boldsymbol{x}$ in Eq. (1) is partitioned into kept and reduced degrees of freedom as follows:

$$\boldsymbol{x} = \begin{matrix} \boldsymbol{x}_k \\ \boldsymbol{x}_r \end{matrix} \tag{7}$$

However, such partitioning of the displacement vector cannot be arbitrary. Some guidelines have been proposed in the literature[9–16] for choosing the degrees of freedom for an accurate representation of the lower modes, which are usually the most important in the dynamic analysis. To decide about the degrees of freedom to be kept, associated with $\boldsymbol{x}_k$, it is common to compute the ratio of the diagonal coefficients of the stiffness and mass matrices K_{ii}/M_{ii} and then select the first n_k degrees of freedom with the smallest ratios as the kept degrees of freedom. The remaining n_r degrees of freedom belong to $\boldsymbol{x}_r$. Of course, $N = n_k + n_r$.

Consistent with the partitioning of the displacement vector, the mass, damping, and stiffness matrices also are partitioned as follows:

$$[M] = \begin{matrix} M_{kk} & M_{kr} \\ M_{rk} & M_{rr} \end{matrix}, \qquad [C] = \begin{matrix} C_{kk} & C_{kr} \\ C_{rk} & C_{rr} \end{matrix}$$

$$[K] = \begin{matrix} K_{kk} & K_{kr} \\ K_{rk} & K_{rr} \end{matrix} \tag{8}$$

in which submatrices $[M_{ij}]$, $[C_{ij}]$, and $[K_{ij}]$ have dimensions $(n_i \times n_j)$ with $i, j = k, r$.

A similar partitioning also is carried out on the state vector and on the matrices of the state equation (2) as

$$\boldsymbol{y} = \begin{matrix} \boldsymbol{y}_k \\ \boldsymbol{y}_r \end{matrix} \tag{9}$$

where

$$\boldsymbol{y}_j = \begin{matrix} \dot{\boldsymbol{x}}_j \\ \boldsymbol{x}_j \end{matrix}, \qquad j = k, r \tag{10}$$

$$[A] = \begin{matrix} A_{kk} & A_{kr} \\ A_{rk} & A_{rr} \end{matrix}, \qquad [B] = \begin{matrix} B_{kk} & B_{kr} \\ B_{rk} & B_{rr} \end{matrix} \tag{11}$$

in which submatrices $[A_{ij}]$ and $[B_{ij}]$ have dimensions $(2n_i \times 2n_j)$ and are defined as

$$[A_{ij}] = \begin{matrix} 0_{ij} & M_{ij} \\ M_{ij} & C_{ij} \end{matrix}, \qquad [B_{ij}] = \begin{matrix} M_{ij} & 0_{ij} \\ 0_{ij} & -K_{ij} \end{matrix}, \qquad i, j = k, r \tag{12}$$

The modal matrices in Eq. (5) also are partitioned consistently as follows:

$$[\Omega] = \begin{matrix} \Omega_k & 0 \\ 0 & \Omega_r \end{matrix}, \qquad [\Psi] = \begin{matrix} \Psi_{kk} & \Psi_{kr} \\ \Psi_{rk} & \Psi_{rr} \end{matrix} \tag{13}$$

The submatrices $[\Omega_k]$, $[\Psi_{kk}]$, and $[\Psi_{rk}]$ contain the eigenvalues and eigenvectors corresponding to the first $2n_k$ lower modes, i.e., those with the smaller absolute values of the associated eigenvalue. Similarly, the submatrices $[\Omega_r]$, $[\Psi_{kr}]$, and $[\Psi_{rr}]$ will contain the remaining $2n_r$ eigenvalues and eigenvectors corresponding to the higher modes. Both sets of modes occur in complex conjugate pairs. The eigenvector submatrices $[\Psi_{ij}]$ are of size $(2n_i \times 2n_j)$ with $i, j = k, r$.

Lower-Mode Condensation

The part of the eigenvalue problem in Eq. (5) corresponding to the $2n_k$ lower modes can be written as

$$\begin{matrix} A_{kk} & A_{kr} & \Psi_{kk} \\ A_{rk} & A_{rr} & \Psi_{rk} \end{matrix} [\Omega_k] = \begin{matrix} B_{kk} & B_{kr} & \Psi_{kk} \\ B_{rk} & B_{rr} & \Psi_{rk} \end{matrix} \tag{14}$$

To reduce the size of this eigenvalue problem, we express matrix $[\Psi_{rk}]$ in terms of matrix $[\Psi_{kk}]$ as indicated by

$$[\Psi_{rk}] = [R_l][\Psi_{kk}] \tag{15}$$

in which $[R_l]$, the condensation matrix associated with the lower modes, is defined as

$$[R_l] = [\Psi_{rk}][\Psi_{kk}]^{-1} \tag{16}$$

Using Eq. (16), the eigenvector matrix of the lower modes can be written in terms of the condensation matrix $[R_l]$ as

$$\begin{matrix} \Psi_{kk} \\ \Psi_{rk} \end{matrix} = \begin{matrix} I_k \\ R_l \end{matrix} [\Psi_{kk}] = [S_l][\Psi_{kk}] \tag{17}$$

By introducing Eq. (17) in Eq. (14) and premultiplying by $[S_l]^T$, the condensed eigenvalue problem associated with the lower modes is obtained as follows:

$$[A_l][\Psi_{kk}][\Omega_k] = [B_l][\Psi_{kk}] \tag{18}$$

where the condensed matrices $[A_l]$ and $[B_l]$ are

$$[A_l] = [A_{kk}] + [R_l]^T[A_{rk}] + [A_{kr}][R_l] + [R_l]^T[A_{rr}][R_l] \tag{19a}$$

$$[B_l] = [B_{kk}] + [R_l]^T[B_{rk}] + [B_{kr}][R_l] + [R_l]^T[B_{rr}][R_l] \tag{19b}$$

Higher-Mode Condensation

One can similarly define the condensed eigenvalue problem associated with the remaining $2n_r$ higher modes as described in the following. The part of the eigenvalue problem in Eq. (5) that corresponds to the $2n_r$ higher modes can be written as

$$\begin{matrix} A_{kk} & A_{kr} & \Psi_{kr} \\ A_{rk} & A_{rr} & \Psi_{rr} \end{matrix} [\Omega_r] = \begin{matrix} B_{kk} & B_{kr} & \Psi_{kr} \\ B_{rk} & B_{rr} & \Psi_{rr} \end{matrix} \tag{20}$$

To reduce this eigenvalue problem, we express matrix $[\Psi_{kr}]$ in terms of $[\Psi_{rr}]$ as follows:

$$[\Psi_{kr}] = [R_h][\Psi_{rr}] \tag{21}$$

where the matrix $[R_h]$, the condensation matrix associated with the higher modes, is defined as

$$[R_h] = [\Psi_{kr}][\Psi_{rr}]^{-1} \tag{22}$$

Using Eq. (22), the eigenvector matrix of the higher modes can be written in terms of the condensation matrix $[R_h]$ as

$$\begin{matrix} \Psi_{kr} \\ \Psi_{rr} \end{matrix} = \begin{matrix} R_h \\ I_r \end{matrix} [\Psi_{rr}] = [S_h][\Psi_{rr}] \tag{23}$$

By introducing Eq. (23) into Eq. (20) and premultiplying by $[S_h]^T$, we obtain the condensed eigenvalue problem associated with the higher modes as

$$[A_h][\Psi_{rr}][\Omega_r] = [B_h][\Psi_{rr}] \tag{24}$$

where the condensed matrices $[A_h]$ and $[B_h]$ are

$$[A_h] = [A_{rr}] + [R_h]^T[A_{kr}] + [A_{rk}][R_h] + [R_h]^T[A_{kk}][R_h] \tag{25a}$$

$$[B_h] = [B_{rr}] + [R_h]^T[B_{kr}] + [B_{rk}][R_h] + [R_h]^T[B_{kk}][R_h] \tag{25b}$$

From the solution of Eqs. (18) and (24), one can obtain the complete set of eigenproperties of the structural system. Once $[\Psi_{kk}]$ and $[\Psi_{rr}]$ are known, submatrices $[\Psi_{rk}]$ and $[\Psi_{kr}]$ can be computed from Eqs. (15) and (21), respectively. It can be shown that the complete set of eigenvectors computed from the reduced eigenvalue problems automatically will be normalized as indicated in the first part of Eq. (6) if the eigenvector submatrices $[\Psi_{kk}]$ and $[\Psi_{rr}]$ are normalized such that

$$[\Psi_{kk}]^T[A_l][\Psi_{kk}] = [I_k] \tag{26a}$$

$$[\Psi_{rr}]^T[A_h][\Psi_{rr}] = [I_r] \tag{26b}$$

The complete eigenvector matrix then is given as

$$[\Psi] = \begin{matrix} \Psi_{kk} & \Psi_{kr} \\ \Psi_{rk} & \Psi_{rr} \end{matrix} = \begin{matrix} \Psi_{kk} & R_h\Psi_{rr} \\ R_l\Psi_{kk} & \Psi_{rr} \end{matrix} \tag{27}$$

Note here that to construct the condensed eigenvalue problems of Eqs. (18) and (24) requires that the condensation matrices $[R_l]$ and $[R_h]$ be known. However, Eqs. (16) and (22) cannot be used because the eigenvectors required in these equations are not known a priori. In the following section, therefore, we present an iterative method to construct these condensation matrices.

Later, we show that these condensation matrices are real, although they are defined in Eqs. (18) and (24) in terms of complex matrices. This fact also ensures that the state matrices $[A_l]$, $[B_l]$, $[A_h]$, and $[B_h]$ in the eigenvalue problems in Eqs. (18) and (24) are all real matrices.

Relationship Between $[R_l]$ and $[R_h]$

It is also of interest to derive the relationship between the two condensation matrices. For this we use the first of the two orthogonality conditions of the eigenvector matrix $[\Psi]$ in Eq. (6). This equation is equivalent to the following four matrix equations:

$$[\Psi_{kk}]^T[A_{kk}\Psi_{kk} + A_{kr}\Psi_{rk}] + [\Psi_{rk}]^T[A_{rk}\Psi_{kk} + A_{rr}\Psi_{rk}] = [I_k] \tag{28a}$$

$$[\Psi_{kk}]^T[A_{kk}\Psi_{kr} + A_{kr}\Psi_{rr}] + [\Psi_{rk}]^T[A_{rk}\Psi_{kr} + A_{rr}\Psi_{rr}] = [0] \tag{28b}$$

$$[\Psi_{kr}]^T[A_{kk}\Psi_{kk} + A_{kr}\Psi_{rk}] + [\Psi_{rr}]^T[A_{rk}\Psi_{kk} + A_{rr}\Psi_{rk}] = [0] \tag{28c}$$

$$[\Psi_{kr}]^T[A_{kk}\Psi_{kr} + A_{kr}\Psi_{rr}] + [\Psi_{rr}]^T[A_{rk}\Psi_{kr} + A_{rr}\Psi_{rr}] = [I_r] \tag{28d}$$

Substituting Eqs. (15) and (21) in Eq. (28b) and rearranging the terms, we obtain

$$[\Psi_{kk}]^T\ A_{kk}R_h + A_{kr} + R_l^T A_{rk}R_h + R_l^T A_{rr}\ [\Psi_{rr}] = [0] \tag{29}$$

For Eq. (29) to be zero, the matrix in the middle must be a null matrix; hence,

$$A_{kk}R_h + A_{kr} + R_l^T A_{rk}R_h + R_l^T A_{rr} = [0] \tag{30}$$

Solving for $[R_h]$, we obtain the desired relationship:

$$[R_h] = -\ A_{kk} + R_l^T A_{rk}\ ^{-1}\ A_{kr} + R_l^T A_{rr} \tag{31}$$

Thus, if one knows the condensation matrix $[R_l]$, Eq. (31) then can be used to compute $[R_h]$ as well. This will be of help in defining the reduced eigenvalue problem for the higher modes if the eigenvalue problem of the lower modes is solved. An equation similar to Eq. (31) can be developed to calculate $[R_l]$ if one knows $[R_h]$. For this purpose we use the second part of Eq. (6). This provides

$$[R_l]^T = -[B_{kk}R_h + B_{kr}][B_{rk}R_h + B_{rr}]^{-1} \tag{32}$$

Note that, to define the matrices of the condensed eigenvalue problems, we have to know at least one of the two condensation matrices $[R_l]$ or $[R_h]$ in advance because the other can be calculated from either Eq. (31) or Eq. (32).

Iterative Condensation Scheme for Lower Modes

An initial approximation for the condensation matrix $[R_l]$ can be obtained from the eigenvalue problem in Eq. (14) associated with the lower modes. This equation is equivalent to the following matrix equations:

$$[A_{kk}\Psi_{kk} + A_{kr}\Psi_{rk}][\Omega_k] = [B_{kk}\Psi_{kk} + B_{kr}\Psi_{rk}] \tag{33a}$$

$$[A_{rk}\Psi_{kk} + A_{rr}\Psi_{rk}][\Omega_k] = [B_{rk}\Psi_{kk} + B_{rr}\Psi_{rk}] \tag{33b}$$

To obtain an initial approximation of the condensation matrix, we assume that the left-hand side of Eq. (33b) is equal to a null matrix. This is equivalent to assuming that the eigenvalues in $[\Omega_k]$ are zero, and it is done only to obtain an initial relationship between the kept and reduced modal coordinates. The equation then is reduced to

$$B_{rk}\Psi_{kk}^{(0)} + B_{rr}\Psi_{rk}^{(0)} = [0] \tag{34}$$

where the superscript (0) indicates that the quantities are associated with the initial estimate. Solving Eq. (34), one obtains

$$\Psi_{rk}^{(0)} = -[B_{rr}]^{-1}[B_{rk}]\ \Psi_{kk}^{(0)} \tag{35}$$

Comparing this equation with Eq. (15), we obtain an initial approximation for the condensation matrix $[R_l]$ as follows:

$$R_l^{(0)} = -[B_{rr}]^{-1}[B_{rk}] \tag{36}$$

Using the second part of Eq. (12), this equation can be written as

$$R_l^{(0)} = \begin{matrix} G_M & 0 \\ 0 & G_K \end{matrix} = - \begin{matrix} M_{rr}^{-1}M_{rk} & 0 \\ 0 & K_{rr}^{-1}K_{rk} \end{matrix} \tag{37}$$

in which submatrices G_M and G_K are akin to Guyan condensation matrices[1] in terms of the mass and stiffness matrices, respectively. There is an alternative initial estimate for the condensation matrix $[R_l^{(0)}]$. It is obtained by replacing submatrix G_M with submatrix G_K in the preceding equation as follows:

$$R_l^{(0)} = \begin{matrix} G_K & 0 \\ 0 & G_K \end{matrix} = - \begin{matrix} K_{rr}^{-1}K_{rk} & 0 \\ 0 & K_{rr}^{-1}K_{rk} \end{matrix} \tag{38}$$

This alternative definition of $[R_l^{(0)}]$ has the advantage that the calculation of the inverse of matrix $[M_{rr}]$ is avoided. In addition, several numerical trials carried out using Eqs. (37) and (38) have shown that the latter leads to better initial values of the eigenproperties sought. With the initial estimate of the condensation matrix given by Eq. (38), it can be shown that matrices $[A_l]$ and $[B_l]$ of the eigenvalue problem in Eq. (18) can be expressed as

$$[A_l] = \begin{matrix} [0] & [M_l] \\ [M_l] & [C_l] \end{matrix}, \qquad [B_l] = \begin{matrix} [M_l] & [0] \\ [0] & -[K_l] \end{matrix} \tag{39}$$

in which the mass $[M_l]$, damping $[C_l]$, and stiffness $[K_l]$ matrices are condensed using the simple Guyan condensation matrix as follows:

$$[M_l] = [M_{kk}] + [G_K]^T[M_{rk}] + [M_{kr}][G_K] + [G_K]^T[M_{rr}][G_K] \tag{40a}$$

$$[C_l] = [C_{kk}] + [G_K]^T[C_{rk}] + [C_{kr}][G_K] + [G_K]^T[C_{rr}][G_K] \tag{40b}$$

$$[K_l] = [K_{kk}] + [G_K]^T[K_{rk}] + [K_{kr}][G_K] + [G_K]^T[K_{rr}][G_K] \tag{40c}$$

With the condensation matrix in Eq. (37) or (38) known, we can compute the matrices of the condensed eigenvalue problem in Eq. (18) associated with the lower modes and then obtain an initial approximation of the eigenproperties $[\Psi_{kk}^{(0)}]$ and $[\Omega_k^{(0)}]$. This approximate solution now can be used to improve matrix $[R_l]$. To define the improved estimate for $[R_l]$, we consider Eq. (33b) again, which, after terms are reordered, is written as

$$[B_{rr}\Psi_{rk}] = [A_{rk}\Psi_{kk} + A_{rr}\Psi_{rk}][\Omega_k] - [B_{rk}\Psi_{kk}] \tag{41}$$

By pre- and postmultiplying Eq. (41) by $[B_{rr}]^{-1}$ and $[\Psi_{kk}]^{-1}$, respectively, and recalling the definition of $[R_l]$ in Eq. (16), we obtain

$$[R_l] = [B_{rr}]^{-1}[A_{rk} + A_{rr}R_l][\Psi_{kk}][\Omega_k][\Psi_{kk}]^{-1} - [B_{rr}]^{-1}[B_{rk}] \tag{42}$$

Equation (42) requires the inverse of the complex matrix $[\Psi_{kk}]$. However, we can use the orthogonality condition of the modes to avoid calculating this inverse. We consider the following orthogonality condition:

$$[\Psi_{kk}]^T[B_l][\Psi_{kk}] = [\Omega_k] \tag{43}$$

If we postmultiply this equation by $[\Psi_{kk}]^{-1}$, we get

$$[\Omega_k][\Psi_{kk}]^{-1} = [\Psi_{kk}]^T[B_l] \tag{44}$$

By introducing Eq. (44) into Eq. (42), we obtain the following expression for $[R_l]$:

$$[R_l] = [B_{rr}]^{-1}[A_{rk} + A_{rr}R_l][\Psi_{kk}][\Psi_{kk}]^T[B_l] - [B_{rr}]^{-1}[B_{rk}] \tag{45}$$

Equation (45) will be satisfied identically if the actual condensation matrix $[R_l]$ defined by Eq. (16) and eigenvector matrix $[\Psi_{kk}]$ are used in the right-hand side of the equation. However, this equation also can be used to define a recursive relationship to iteratively improve $[R_l]$ as follows:

$$R_l^{(i+1)} = [B_{rr}]^{-1}\left[A_{rk} + A_{rr}R_l^{(i)}\right]\left[\Psi_{kk}^{(i)}\right]\left[\Psi_{kk}^{(i)}\right]^T\left[B_l^{(i)}\right] - [B_{rr}]^{-1}[B_{rk}] \tag{46}$$

The last term on the right-hand side of this equation can be identified as the initial approximation $[R_l^{(0)}]$. The other term is a correction term that is added to improve this original estimate. Once again this improved condensation matrix is defined in terms of complex eigenvectors, but we show next that it is always a real matrix.

Consider the case when $i = 0$. From Eq. (46), we can see that the first estimate of the condensation matrix $[R_l^{(1)}]$ will be real if the term

$$\Psi_{kk}^{(0)}\ \Psi_{kk}^{(0)\ T} \tag{47}$$

is real. To prove this, let us consider the orthogonality condition

$$\Psi_{kk}^{(0)\ T}\ A_l^{(0)}\ \Psi_{kk}^{(0)} = [I_k] \tag{48}$$

By pre- and postmultiplying this equation by $\Psi_{kk}^{(0)\ -T}$ and $\Psi_{kk}^{(0)\ -1}$, respectively, and then computing its inverse, we get

$$\Psi_{kk}^{(0)}\ \Psi_{kk}^{(0)\ T} = A_l^{(0)\ -1} \tag{49}$$

where the matrix $[A_l^{(0)}]$ is real because the initial condensation matrix is real. Thus, $[R_l^{(1)}]$ defined from Eq. (46) will be real, which in turn renders the remaining matrices $[A_l^{(1)}]$ and $[B_l^{(1)}]$ real. Thus, matrices $[R_l^{(i)}]$, $[A_l^{(i)}]$, and $[B_l^{(i)}]$ involved in all subsequent iterations also will be real. This also will apply to the condensation matrix for the higher modes $[R_h^{(i)}]$, to be defined later in Eq. (60).

The improved condensation matrix $R_l^{(i+1)}$ can be used to define a new condensed eigenvalue problem in Eq. (18). The eigenvector matrix obtained from the solution of this eigenvalue problem then is used to calculate a new condensation matrix by using Eq. (46). The procedure is repeated until the desired convergence criterion on the eigenvalues at two consecutive iterations steps is achieved.

After the lower modes are obtained, the condensation matrix $[R_l]$ from the final iteration step can be used in Eq. (31) to calculate the condensation matrix for the higher modes. With matrix $[R_h]$ known, the higher modes can be obtained from the solution of their condensed eigenvalue problem in Eq. (24). The complete set of eigenvectors then is computed with Eq. (27).

The foregoing procedure allows us to compute the complete set of eigenproperties of a structural system by iteratively solving the eigenvalue problem associated with the lower modes. Furthermore, as we show in the next section, it is also possible to compute the eigenproperties by iterating with the higher modes.

Iterative Condensation Scheme for Higher Modes

The eigenvalue problem for the remaining higher modes, defined in Eq. (20), is equivalent to the following matrix equations:

$$[A_{kk}\Psi_{kr} + A_{kr}\Psi_{rr}][\Omega_r] = [B_{kk}\Psi_{kr} + B_{kr}\Psi_{rr}] \tag{50a}$$

$$[A_{rk}\Psi_{kr} + A_{rr}\Psi_{rr}][\Omega_r] = [B_{rk}\Psi_{kr} + B_{rr}\Psi_{rr}] \tag{50b}$$

To obtain an initial estimate for the condensation matrix $[R_h]$, we use Eq. (50a) with its right-hand side set equal to the null matrix to obtain

$$A_{kk}\Psi_{kr}^{(0)} + A_{kr}\Psi_{rr}^{(0)} = [0] \tag{51}$$

Solving the preceding equation for $[\Psi_{kr}^{(0)}]$, we get

$$\Psi_{kr}^{(0)} = -[A_{kk}]^{-1}[A_{kr}]\ \Psi_{rr}^{(0)} \tag{52}$$

Comparing this equation with Eq. (21), we obtain the initial approximation for $[R_h]$ as

$$R_h^{(0)} = -[A_{kk}]^{-1}[A_{kr}] \tag{53}$$

or in terms of the mass and damping submatrices as

$$R_h^{(0)} = \begin{matrix} M_{kk}^{-1}M_{kr} & M_{kk}^{-1}\ C_{kk}M_{kk}^{-1}M_{kr} + C_{kr} \\ 0 & M_{kk}^{-1}M_{kr} \end{matrix} \tag{54}$$

where the superscript (0) indicates that it is an initial value. Once the initial condensation matrix is known, we can compute the matrices of the condensed eigenvalue problem associated with the higher modes, Eq. (24). The solution of this eigenvalue problem provides an initial approximation of the eigenproperties $[\Psi_{rr}^{(0)}]$ and $[\Omega_r^{(0)}]$.

We proceed now to develop the recursive relationship to improve the initial estimate of $[R_h]$.

To develop a recursive equation similar to Eq. (46) obtained for the lower modes, consider again Eq. (50a). Solving for $[A_{kk}\Psi_{kr}]$ and substituting Eq. (21) into the right-hand side of the resulting expression leads to

$$[A_{kk}\Psi_{kr}] = [B_{kk}R_h + B_{kr}][\Psi_{rr}][\Omega_r]^{-1} - [A_{kr}\Psi_{rr}] \tag{55}$$

Premultiplying Eq. (55) by $[A_{kk}]^{-1}$ and postmultiplying it by $[\Psi_{rr}]^{-1}$, and taking into account the definition of $[R_h]$ in Eq. (22), we obtain

$$[R_h] = [A_{kk}]^{-1}[B_{kk}R_h + B_{kr}][\Psi_{rr}][\Omega_r]^{-1}[\Psi_{rr}]^{-1} - [A_{kk}]^{-1}[A_{kr}] \tag{56}$$

The inverse of $[\Psi_{rr}]$ in this equation also can be obtained from the orthogonality condition

$$[\Psi_{rr}]^T[A_h][\Psi_{rr}] = [I_r] \tag{57}$$

By postmultiplying this equation by $[\Psi_{rr}]^{-1}$, we obtain

$$[\Psi_{rr}]^{-1} = [\Psi_{rr}]^T[A_h] \tag{58}$$

Introducing this equation into Eq. (56), we can write the condensation matrix as

$$[R_h] = [A_{kk}]^{-1}[B_{kk}R_h + B_{kr}][\Psi_{rr}][\Omega_r]^{-1}[\Psi_{rr}]^T[A_h] - [A_{kk}]^{-1}[A_{kr}] \tag{59}$$

Equation (59) will be satisfied if the matrix $[R_h]$ defined by Eq. (22) is used in the right-hand side. However, as in the case of the iterative procedure with lower modes, this equation also can be used to define the recursive equation to improve the estimate of the condensation matrix $[R_h]$ as follows:

$$R_h^{(i+1)} = [A_{kk}]^{-1}\ R_{kr}^{(i)}\ \ \Psi_{rr}^{(i)}\ \ \Omega_r^{(i)\ -1}\ \Psi_{rr}^{(i)\ T}\ A_h^{(i)} - [A_{kk}]^{-1}[A_{kr}] \tag{60}$$

where

$$R_{kr}^{(i)} = B_{kk}R_h^{(i)} + B_{kr} \tag{61}$$

As in Eq. (46), the last term on the right-hand side of Eq. (60) can be identified as the initial approximation of the condensation matrix. The other term is a correction term that is added to the initial approximation to improve it.

If the condensation matrix $[R_h]$ is known, then the iteration with the higher modes can be performed. In each iteration, the eigenvalue problem in Eq. (24) is solved. This solution then is used to improve the estimate of the condensation matrix by means of Eqs. (60) and (61). The procedure continues until the desired convergence criterion is satisfied. If desired, the condensation matrix for the lower modes is computed using Eq. (32) and the lower modes can be computed from the solution of the eigenvalue problem in Eq. (18).

Numerical Results

To demonstrate the application and convergence properties of the proposed dynamic condensation approach, two examples have been considered. The first example is the clamped-clamped beam shown in Fig. 1, used earlier by Craig and Chung.[17] Although it is a small-dimension problem for which one normally would not apply a condensation procedure, it is considered primarily to present the numerical results in a manageable tabular form to show the convergence trends and the errors one can expect in the calculations. Later, the numerical results of a somewhat larger truss problem, shown in Fig. 2, also are presented to demonstrate the effect on the convergence of the method used for choosing the degrees of freedom to be condensed.

To construct a nonclassical damped system, the beam structure is divided into two substructures, α and β. Next it is assumed that the damping matrices in the two parts are proportional to their respective stiffness matrices using different proportionality constants as follows:

$$[C_\alpha] = \tfrac{1}{48}[K_\alpha], \qquad [C_\beta] = \tfrac{1}{96}[K_\beta] \tag{62}$$

where $[C_j]$ and $[K_j]$ are, respectively, the damping and stiffness matrices of substructure j. The structure is discretized using 10 beam finite elements. The beam has a total of 18 degrees of freedom with 2 degrees of freedom per node: a vertical displacement and a rotation.

The complex eigenvalues of this clamped-clamped beam calculated without any condensation are given in Table 1 and are considered the exact values for comparison purposes. Because all eigenvalues occur in complex conjugate pairs, only one value of each pair is given. The eigenvalue corresponding to the jth mode is denoted as

$$\lambda_j = -\sigma_j + i\theta_j \tag{63}$$

in which σ_j and θ_j are the real and imaginary parts, respectively. The equivalent modal frequencies and damping ratios for each mode also are given in Table 1 and are computed as

$$\omega_j = \sqrt{\sigma_j^2 + \theta_j^2}, \qquad \beta_j = \sigma_j/\omega_j \tag{64}$$

Table 1 Exact eigenvalues, modal frequencies, and damping ratios of the clamped-clamped beam in Fig. 1

Mode	Eigenvalue	Modal frequency, rad/s	Damping ratio
1	$-0.0004 + i0.2237$	0.224	0.0019
2	$-0.0030 + i0.6169$	0.617	0.0049
3	$-0.0125 + i1.2102$	1.210	0.0103
4	$-0.0332 + i2.0037$	2.004	0.0166
5	$-0.0747 + i3.0018$	3.003	0.0249
6	$-0.1498 + i4.2127$	4.215	0.0355
7	$-0.2630 + i5.6458$	5.652	0.0465
8	$-0.4469 + i7.3033$	7.317	0.0611
9	$-0.6935 + i9.0750$	9.101	0.0762
10	$-1.2359 + i12.0952$	12.158	0.1017
11	$-1.8204 + i14.6402$	14.753	0.1234
12	$-2.6403 + i17.7727$	17.968	0.1470
13	$-4.0126 + i21.3880$	21.761	0.1844
14	$-5.7159 + i25.8828$	26.506	0.2156
15	$-8.1141 + i30.2511$	31.320	0.2591
16	$-13.0760 + i36.1990$	38.488	0.3397
17	$-12.4335 + i40.7452$	42.600	0.2919
18	$-22.2042 + i41.4118$	46.989	0.4725

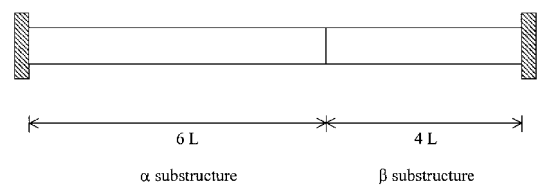

Fig. 1 Structural configuration of the clamped-clamped beam used in the first numerical example.

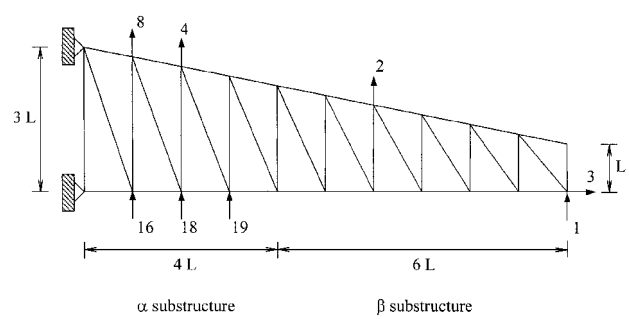

Fig. 2 Structural configuration of the truss used in the second numerical example showing the kept degrees of freedom numbered according to Shah and Raymund's[12] procedure.

The results obtained with the condensation approach are evaluated using measures of eigenvalues and eigenvectors errors. The measure of the eigenvalue errors is obtained by comparing the exact modal frequencies and damping ratios with those obtained with the condensation approach. The relative error ϵ_ω for modal frequencies is defined as

$$\epsilon_\omega = \frac{(\omega_{\text{appr}} - \omega_{\text{exact}})}{\omega_{\text{exact}}} \times 100 \tag{65}$$

Relative errors for the damping ratios, ϵ_β, are computed in a similar way. For the eigenvectors, a percent rms error is used to indicate how well they compare with the exact ones. The rms error is defined as

$$\text{rms error} = \frac{\|\psi_{\text{appr}} - \psi_{\text{exact}}\|_2}{\|\psi_{\text{exact}}\|_2} \times 100 \tag{66}$$

where the 2-norm of the complex vector ψ is defined as

$$\|\psi\|_2 = \sqrt{\sum_{j=1}^{n} |\text{Re}(\psi_j)|^2 + |\text{Im}(\psi_j)|^2} \tag{67}$$

The displacement vector of the clamped-clamped beam has been partitioned into two sets of nine degrees of freedom each. Translational degrees of freedom were kept to formulate the eigenvalue problem associated with the lower modes, whereas the rotational degrees of freedom were used to compute the higher modes. At each iteration step, the whole set of modes is computed from the solution of the eigenvalue problems in Eqs. (18) and (24). Then the condensation matrix is updated with Eq. (46) when iterating with the lower modes and with Eq. (60) when iterating with the higher modes.

Tables 2–4 show the results obtained when iterating with the lower modes. The initial condensation matrix $[R_l^{(0)}]$ was evaluated using Eq. (38). A dash in columns of these tables, and in other tables to follow, indicates that the error is less than 0.0005% and the value is regarded as exact. It is seen that two iterations are enough to obtain all of the modal frequencies and damping ratios with absolute errors less than 1%. After four iterations, 15 frequencies and 12 damping ratios are exact. Nonetheless, as shown in Table 4, the rms errors in several eigenvectors are considerably higher at the beginning of the iteration process, but after four iterations only three of them are greater than 1%, with 4.7% being the largest. Tables 5–7 show the results obtained when iterating with the higher modes. In this case, four iterations are required to reduced the absolute errors below 1% in all modal frequencies and damping ratios. Twelve frequencies and ten damping ratios are exact and five eigenvectors have rms errors greater than 1% after four iterations. Also, a mutual comparison of the errors associated with the lower- and higher-mode

Table 2 Percent error in modal frequencies of clamped-clamped beam in Fig. 1 calculated by iterating with lower modes

	Iteration step				
Mode	0	1	2	3	4
1	0.000	—	—	—	—
2	0.003	0.000	—	—	—
3	0.029	0.000	—	—	—
4	0.145	0.000	—	—	—
5	0.518	0.001	0.000	—	—
6	1.427	0.003	0.000	—	—
7	3.070	0.033	0.000	—	—
8	4.411	0.364	0.044	0.003	0.001
9	1.725	0.201	0.100	0.018	0.013
10	−0.806	−0.047	−0.023	−0.020	−0.012
11	−1.137	−0.118	−0.018	−0.002	0.000
12	−0.369	−0.003	−0.001	0.000	—
13	−0.087	0.000	—	—	—
14	−0.023	0.000	—	—	—
15	−0.005	0.000	—	—	—
16	−0.001	0.000	—	—	—
17	−0.003	0.000	—	—	—
18	0.000	—	—	—	—

Table 3 Percent error in damping ratios of clamped-clamped beam in Fig. 1 calculated by iterating with lower modes

	Iteration step				
Mode	0	1	2	3	4
1	0.004	0.000	—	—	—
2	−0.043	0.000	—	—	—
3	0.064	0.000	—	—	—
4	−0.227	0.004	0.000	—	—
5	−0.694	0.039	0.000	—	—
6	−0.346	0.260	−0.001	0.000	—
7	−5.444	1.284	0.026	0.001	0.000
8	0.191	3.684	0.602	0.196	0.046
9	−4.760	0.803	0.100	0.051	0.038
10	−0.377	−0.840	−0.139	−0.055	−0.026
11	2.316	−0.476	−0.029	0.021	−0.003
12	0.062	−0.192	−0.012	−0.002	−0.001
13	0.109	−0.029	0.000	—	—
14	0.005	−0.003	0.000	—	—
15	0.020	−0.001	0.000	—	—
16	−0.007	0.000	—	—	—
17	0.005	0.000	—	—	—
18	0.001	0.000	—	—	—

Table 4 Percent rms error in eigenvectors of clamped-clamped beam in Fig. 1 calculated by iterating with lower modes

	Iteration step				
Mode	0	1	2	3	4
1	0.010	0.000	—	—	—
2	0.132	0.002	0.000	—	—
3	0.663	0.014	0.001	0.000	—
4	2.164	0.086	0.005	0.001	0.000
5	5.559	0.368	0.022	0.004	0.002
6	12.434	1.386	0.138	0.043	0.019
7	24.557	4.995	0.824	0.246	0.122
8	42.399	16.997	6.672	3.352	1.595
9	44.783	20.422	10.172	6.553	4.767
10	9.355	4.289	2.177	1.403	1.084
11	8.118	3.018	1.047	0.517	0.205
12	3.411	1.250	0.338	0.167	0.107
13	1.431	0.471	0.091	0.072	0.050
14	0.520	0.204	0.025	0.024	0.023
15	0.369	0.091	0.019	0.017	0.013
16	0.204	0.038	0.010	0.006	0.004
17	0.238	0.073	0.013	0.006	0.005
18	0.129	0.013	0.004	0.003	0.002

Table 5 Percent error in modal frequencies of clamped-clamped beam in Fig. 1 calculated by iterating with higher modes

	Iteration step				
Mode	0	1	2	3	4
1	0.020	0.000	—	—	—
2	0.095	0.000	—	—	—
3	0.363	0.000	—	—	—
4	0.899	0.003	0.000	—	—
5	1.918	0.027	0.001	0.000	—
6	3.495	0.152	0.001	0.000	—
7	5.277	0.614	0.063	0.013	0.002
8	5.721	1.543	0.454	0.164	0.061
9	2.030	0.675	0.226	0.077	0.024
10	−3.153	−1.302	−0.556	−0.220	−0.081
11	−6.582	−1.727	−0.668	−0.238	−0.090
12	−5.486	−0.699	−0.170	−0.028	−0.005
13	−3.690	−0.211	−0.025	−0.001	0.000
14	−2.200	−0.053	−0.002	0.000	—
15	−1.016	−0.012	0.000	—	—
16	−0.026	−0.001	0.000	—	—
17	−0.641	−0.004	0.000	—	—
18	−0.088	0.000	—	—	—

Table 6 Percent error in damping ratios of clamped-clamped beam in Fig. 1 calculated by iterating with higher modes

	Iteration step				
Mode	0	1	2	3	4
1	0.022	0.000	—	—	—
2	0.113	0.000	—	—	—
3	−0.032	0.001	0.000	—	—
4	−0.290	0.009	0.006	0.000	—
5	−1.099	0.027	0.044	0.001	0.000
6	−2.019	0.031	0.241	0.024	0.003
7	4.972	−0.102	0.941	0.301	0.077
8	−1.821	0.505	1.627	1.071	0.598
9	−4.345	−1.425	−0.064	0.128	0.105
10	0.682	−0.123	−0.756	−0.657	−0.406
11	8.102	1.859	−0.554	−0.426	−0.237
12	3.627	0.215	−0.586	−0.180	−0.047
13	3.429	0.062	−0.156	−0.020	−0.002
14	1.039	−0.015	−0.029	−0.002	0.000
15	0.899	0.003	−0.004	0.000	—
16	−0.243	−0.002	−0.001	0.000	—
17	0.505	−0.001	−0.001	0.000	—
18	0.065	0.000	—	—	—

Table 7 Percent rms error in eigenvectors of clamped-clamped beam in Fig. 1 calculated by iterating with higher modes

	Iteration step				
Mode	0	1	2	3	4
1	0.126	0.002	0.001	0.000	—
2	0.660	0.012	0.016	0.004	0.000
3	1.503	0.091	0.025	0.003	0.002
4	3.075	0.415	0.108	0.032	0.024
5	5.589	1.445	0.416	0.068	0.025
6	10.293	4.311	1.490	0.360	0.147
7	19.642	11.467	5.126	2.041	1.141
8	34.777	27.064	15.767	9.784	7.038
9	33.473	28.719	17.831	11.517	8.487
10	18.338	16.885	7.358	7.022	3.667
11	19.975	18.259	6.094	5.667	2.736
12	15.709	11.882	2.974	2.343	0.955
13	10.256	7.917	1.186	0.693	0.197
14	5.763	5.177	0.406	0.182	0.042
15	3.157	2.888	0.140	0.050	0.009
16	2.603	0.693	0.016	0.006	0.001
17	2.913	1.311	0.037	0.013	0.003
18	0.456	0.383	0.011	0.002	0.000

condensations indicates that the iterative process with lower modes provides better initial estimate and convergence than the iterative process with higher modes.[18]

The observations mentioned in this paragraph apply to both lower- and higher-mode iteration cases. All modal frequencies calculated from the eigenvalue problem in Eq. (18) have positive errors in every iteration step, which means that the calculated lower frequencies at each step are higher than the exact ones and that their convergence is monotonically from above. On the other hand, all of the modal frequencies calculated from the eigenvalue problem in Eq. (24) have negative errors, implying that the calculated frequencies are lower than the exact frequencies, and that their convergence is monotonic from below. A similar observation was made for undamped systems by Suarez and Singh.[8] Unlike the errors in the frequencies, the errors in the damping ratios calculated from either Eq. (18) or Eq. (24) are not monotonic and may change sign as the iterative process proceeds. Eigenvectors have larger absolute errors than the errors in the corresponding frequencies and damping ratios. Furthermore, the errors in the modal quantities (frequencies, damping ratios, and eigenvectors) calculated from Eq. (18) increase with the mode number (as indicated by the values in the upper parts of Tables 2–7), whereas the errors in the modal quantities calculated from Eq. (23) decrease with the mode number (as indicated by the values in the lower parts of Tables 2–7).

It is also of interest to examine the accuracy of the eigenproperties obtained when different sets of coordinates are used. The proper selection of coordinates to be kept in the reduced eigenvalue problem will affect the initial estimate of the condensation matrices and this, in turn, also will affect the convergence rate of the iterative process. To demonstrate this, the eigenproperties of the truss structure shown in Fig. 2 have been obtained with different sets of coordinates. Only the iteration process with lower modes is presented. To have a nonclassically damped system, the truss has been divided into two substructures with the damping matrices defined as

$$[C_\alpha] = \tfrac{3}{40}[K_\alpha], \qquad [C_\beta] = \tfrac{1}{40}[K_\beta] \tag{68}$$

Table 8 First 10 exact eigenvalues, modal frequencies, and damping ratios of the truss structure in Fig. 2

Mode	Eigenvalue	Modal frequency, rad/s	Damping ratio
1	$-0.0024 + i2.2760$	0.0228	0.0011
2	$-0.0247 + i5.7412$	0.0574	0.0043
3	$-0.0281 + i9.2934$	0.0929	0.0030
4	$-0.0876 + i11.534$	0.1153	0.0076
5	$-0.1809 + i17.470$	0.1747	0.0136
6	$-0.3397 + i23.529$	0.2353	0.0144
7	$-0.1872 + i26.176$	0.2618	0.0072
8	$-0.5136 + i29.769$	0.2977	0.0173
9	$-0.5461 + i32.438$	0.3244	0.0141
10	$-0.8177 + i36.778$	0.3679	0.0222

The truss consists of 41 elements and 22 joints. Each joint has 2 degrees of freedom, and the structure has a total 40 degrees of freedom. Table 8 shows the first 10 exact eigenvalues, modal frequencies, and damping ratios of the truss, obtained for the full system without any condensation. Once again, only one value of each conjugate pair is given.

Two different ways of selecting the coordinates have been considered. In the first case, the degrees of freedom are ordered according to the magnitude of the stiffness-to-mass ratios K_{ii}/M_{ii}, with the smallest ratios located at the first positions. The first n_k degrees of freedom are kept in this case. In the second case, the degrees of freedom are ordered according to Shah and Raymund's procedure.[12] The procedure is described briefly by the following steps:

1) Find a degree of freedom for which the stiffness-to-mass ratio is the largest.

2) Eliminate this degree of freedom from mass and stiffness matrices by the Guyan reduction method.

3) Apply steps 1 and 2 to the reduced matrices obtained in step 2.

4) Repeat steps 1 to 3 until n_r degrees of freedom are eliminated. The remaining n_k degrees of freedom will be kept.

The degrees of freedom shown in Fig. 2 are numbered according to Shah and Raymund's procedure[12] and are used as kept coordinates. The degrees of freedom indicated as 8, 16, 18, and 19 were those with the smallest K_{ii}/M_{ii} ratios, and thus these were kept in the first case, whereas the degrees of freedom of the truss, numbered 1–4, were those retained in the second case. In both cases, 90% of the coordinates were thus condensed.

The numerical results for the first case are given in Table 9 and for the second case are given in Table 10. In the first case, the initial estimates of the modal frequencies and damping ratios shown in Table 9 are obviously unacceptable. The errors in the initial estimates of modal frequencies range from 56 to 385% and from 224 to 1406% for damping ratios. However, a radical improvement is observed after the first iteration, although the errors for two of the modes are still unacceptable. After four iterations the third mode has the largest error, 5.5% in the modal frequencies and 39% in the damping ratio. A similar behavior is observed in the errors of the eigenvectors. The second choice of selecting the degrees of freedom seems to provide better results, as can be seen in Table 10. The errors in the initial estimates of the modal frequencies range from 0.8 to 18.2% and the errors in the damping ratios from 1.9 to 13.8%. Once again, the errors are reduced considerably after one iteration, and after four iterations all complex modes are very close to the exact ones. Comparing the results in these tables, we can see that the proper selection of coordinates will indeed affect the convergence rate of the condensation approach. Nonetheless,

Table 9 Errors in eigenproperties of the truss in Fig. 2 calculated by iterating with lower modes and with kept degrees of freedom selected according to the stiffness-to-mass ratio

	Iteration step				
Mode	0	1	2	3	4
	Modal frequencies, % errors				
1	56.552	0.040	0.014	0.012	0.011
2	151.611	1.883	0.600	0.505	0.451
3	345.142	34.874	13.776	7.869	5.492
4	385.783	121.935	7.539	4.481	3.712
	Damping ratios, % errors				
1	224.04	−0.42	−0.760	−0.594	−0.506
2	260.34	3.18	−0.976	−1.076	−1.073
3	1406.84	123.19	−63.803	−50.471	−39.309
4	683.85	−7.42	−23.630	−15.778	−12.363
	Eigenvectors, % errors				
1	44.603	0.588	0.155	0.122	0.107
2	76.181	12.329	3.505	2.540	2.097
3	105.775	82.368	24.937	15.707	12.175
4	74.939	61.530	27.180	16.908	13.177

Table 10 Errors in eigenproperties of the truss in Fig. 2 calculated by iterating with lower modes and with kept degrees of freedom selected according to Shah and Raymund's method[12]

	Iteration step				
Mode	0	1	2	3	4
	Modal frequencies, % errors				
1	0.818	0.000	—	—	—
2	6.852	0.047	0.003	0.000	—
3	18.191	0.239	0.028	0.004	0.001
4	7.538	0.044	0.003	0.000	—
	Damping ratios, % errors				
1	−2.007	−0.010	−0.000	—	—
2	1.947	−1.020	−0.052	−0.005	−0.000
3	−13.784	−3.700	−0.562	−0.116	0.001
4	6.557	−1.515	0.044	0.042	0.000
	Eigenvectors, % rms errors				
1	1.286	0.009	0.001	0.000	—
2	11.135	1.128	0.216	0.052	0.140
3	27.400	3.556	1.155	0.482	0.199
4	16.638	1.531	0.736	0.407	0.093

the iterative procedure is able to reduce the errors significantly in all of the eigenproperties no matter what coordinates are selected. If we continue the iteration process, we will eventually obtain exact values for all complex modes in the reduced eigenvalue problem.

Conclusions

The paper presents a dynamic condensation approach for calculating the eigenproperties of nonclassically damped structures. The approach is iterative and can be carried out with the lower or higher modes. Iteration with lower modes, however, provides more accurate results for the same number of iterations. The iteration process starts with the condensation of a selected number of degrees of freedom and an appropriate trial condensation matrix. Guyan's condensation matrix can be used as the trial matrix when seeking the lower modes first. A reduced-size eigenvalue problem is formed using the condensation matrix. The solution of the condensed eigenvalue provides the information required to update the condensation matrix, which in turn is used to define a better-condensed eigenvalue problem. Successive iterations provide better results for the eigenproperties. Usually a few iteration steps provide excellent results. The final condensation matrix can be used to define another complementary eigenvalue problem and the solution of this problem provides the remaining modes.

Two numerical examples are presented to show the effectiveness of the proposed approach. The convergence properties of the method are studied in the first example. The second example is presented to demonstrate the effect on the convergence of choosing different kept degrees of freedom. A careful selection of coordinates can improve the convergence to the final results. That is, with a better selection of the coordinates, fewer iteration steps are required to achieve a certain level of accuracy in the calculated eigenproperties. The modes in the numerical examples considered were all underdamped, i.e., the corresponding eigenvalues had nonzero imaginary parts. When some of the modes are overdamped, some numerical difficulties could arise in the convergence of the method. This situation will be addressed in future studies.

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A. Berman
Associate Editor